ON ASYMPTOTIC EXPANSIONS OF SOLUTIONS OF A CERTAIN CLASS OF INTEGRAL EQUATIONS

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A number of problems in mathematical physics (for example, certain electrostatic problems [1], problems in the theory of radiation, etc.) lead to the solution of singular integral equations of the type

$$a(x)\ \varphi\ (x) = f(x) + \int_0^\infty k\ \left(\frac{x}{y}\right) x^\alpha y^\beta\ \varphi\ (y) dy \tag{1}$$

where $a(x)$ is a finite sum

$$a(x) = \sum_k a_k\ x^{\gamma k} \qquad (a_k = \text{const}) \tag{2}$$

and, if need be, the integral appearing in Equation (1) is to be understood in the Cauchy principal-value sense.

Frequently, it is necessary to obtain only the asymptotic behavior of the solution for large values of x. Let us suppose that the asymptotic expansion is of the form

$$\varphi\ (x) \simeq \sum_{n=0}^\infty \frac{c_n}{x^{an+b}} \qquad (a > 0) \tag{3}$$

If the kernel $k(x)$ does not decrease faster than an arbitrary power of x as x approaches infinity, the determination of an asymptotic expansion of the type of (3) for $\phi(x)$ does not present any particular difficulty. In what follows we shall outline, in a number of cases, a method which furnishes readily asymptotic expansions for solutions of the integral equation (1).

By a Mellin transformation, Equation (1) is transformed into (here, as well as in the following, the conditions for the applicability of the Mellin transform will be supposed to be satisfied, see [2])

$$\sum_k a_k \Phi(s+\gamma_k) = F(s) + K(s+\alpha)\Phi(s+\alpha+\beta+1) \tag{4}$$

where we have employed the notation

$$\Phi(s) = \int_0^\infty \varphi(x) x^{s-1}\, dx \tag{5}$$

(and similar notations for the remaining functions). The difference equation (4) holds in a certain strip of the complex plane, $s_1 < \operatorname{Re} s < s_2$. Without loss of generality, we may suppose that $0 < \operatorname{Re} s < \sigma$, where $\sigma = s_2 - s_1 > 0$.

Let us introduce, instead of the unknown function $\Phi(s)$, a new unknown function $\Psi(s)$, by putting

$$\Phi(s+\delta) = \Omega(s)\Psi(s) \quad (\operatorname{Re} s > 0) \tag{6}$$

where δ is the smallest of the numbers γ_k and $\alpha + \beta + 1$. Equation (2) becomes

$$\sum_k a_k \Omega(s+\gamma_k-\delta)\Psi(s+\gamma_k-\delta) = F(s) + G(s)\Psi(s+\alpha+\beta-\delta+1) \tag{7}$$

where we have used the abbreviation

$$G(s) \equiv K(s+\alpha)\Omega(s+\alpha+\beta-\delta+1) \tag{8}$$

The function $\Omega(s)$ is to be chosen in such a way that the following conditions are satisfied.

In the first place, the function $\Omega(s)$, and also the functions $\Omega(s + \gamma_k - \delta)$ and $G(s)$, which appear as coefficients in Equation (7), must possess inverse Mellin transforms; that is to say, the following integral must exist:

$$\omega(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Omega(s)\frac{ds}{x^s} \qquad (0 < c < \sigma)$$

as well as the corresponding integrals for the functions $\Omega(s + \gamma_k - \delta)$ and $G(s)$.

In the second place, the inverse Mellin transform of all functions occurring in the computation must decrease exponentially as x tends to infinity. In order for this requirement to be fulfilled the functions $\Omega(s)$, $\Omega(s + \gamma_k - \delta)$, and $G(s)$ must be free of singularities in the half-plane $\operatorname{Re} s > 0$. Apparently, the function $\Psi(s)$ may be sought in the form of an asymptotic expansion of the type (3). Recurrence relations for the coefficients c_n, and also the values of the constants a and b, in the asymptotic expansion for $\Psi(x)$ may be obtained from (7). Finally, the unknown function $\Phi(x)$ is easily obtained from (6).

The choice of the function $\Omega(s)$, satisfying the above-mentioned conditions, is not uniquely determined, albeit the relation involving $\Omega(s)$, $G(s)$ and $\Psi(s)$ is such that the final solution $\Phi(x)$ is uniquely determined. In concrete instances of the integral equation (1) it is not difficult to construct functions $\Omega(s)$ which fulfill the desired analytic requirements.

It should be noticed that all that has been said carries over immediately to a wide class of equations which contain Equation (1) as a special case. In the first place, $a(x)$ may be a linear differential operator of the form (see [2])

$$a(x) = \sum_k a_k x^{\gamma_k} \frac{d^{n_k}}{dx^{n_k}}$$

that is to say, Equation (1) may be an integro-differential equation. In the second place, kernels of the following type are also admissible:

$$\begin{array}{ll} x^{\alpha_1} y^{\beta_1} k_1(x / y) + c_1 x^{\gamma_1} y^{\delta_1} & \text{for } x > y \\ & \qquad (\gamma_2 > \gamma_1) \\ x^{\alpha_2} y^{\beta_2} k_2\ (x / y) + c_2 x^{\gamma_2} y^{\delta_2} & \text{for } x < y \end{array}$$

As an illustrative example, let us consider the integral equation

$$\int_0^x \frac{\varphi(y)}{(x-y)^{2/3}}\, dy = q x^{2/3} - \int_0^\infty P(x, y)\, \varphi(y)\, dy, \qquad P(x, y) = \begin{cases} y^{4/3} & \text{for } y < x \\ x^{4/3} & \text{for } y > x \end{cases} \tag{9}$$

This equation (9) arises in the consideration of the temperature distribution in the boundary layer in the flow past a semi-infinite thin plate possessing internal heat sources [3]. Since the boundary-layer equations are strictly applicable only at a large distance from the thin plate, that is, in the domain $x \gg 1$ (the variable x is proportional to the ratio of the distance from the plate and the thickness of the plate), it is only necessary to determine the asymptotic solution of Equation (9) for large values of x.

For brevity, in order not to introduce the generalized (half-plane) Mellin transformation, let us rewrite the nonhomogeneous term in Equation (9), for example in the form

$$\lim_{\varepsilon \to 0} q x^{2/3} e^{-\varepsilon / x}$$

and let us remember to pass to the limit, as $\epsilon \to 0$, after the solution has been carried out.

After this manner of rewriting, from the rewritten equation (9) we obtain

$$\Gamma(^1/_3)\frac{\Gamma(2-s)}{\Gamma(^7/_3-s)}\Phi(s-1)=\lim_{\varepsilon\to 0} q\varepsilon^{s-^2/_3}\Gamma(^2/_3-s)+\frac{4}{3}\frac{1}{s(s-^4/_3)}\Phi(s+1)$$
$$(0<\operatorname{Re} s<^2/_3) \tag{10}$$

where $\Gamma(s)$ is Euler's gamma function.

In accordance with (6), let us introduce a new unknown function $\Psi(s)$ by means of the equation

$$\Phi(s-1)=\frac{\Gamma(2s)\Gamma(s+1)}{\Gamma(2-s)}\Gamma\left(s+\frac{1}{3}\right)\Psi(s) \qquad (\operatorname{Re} s>0) \tag{11}$$

Substituting from (11) into (10), we obtain the following equation for $\Psi(s)$:

$$\Gamma(^1/_3)G_1(s)\Psi(s)=-\lim_{\varepsilon\to 0} q\varepsilon^{s-^2/_3}(s-^4/_3)\Gamma(^2/_3-s)+\frac{4}{3}G_2(s)\Psi(s+2)$$
$$(0<\operatorname{Re} s<^2/_3) \tag{12}$$

where the following abbreviations have been employed:

$$G_1(s)=\frac{\Gamma(2s)\Gamma(s+^1/_3)}{\Gamma(^4/_3-s)}\Gamma(s+1), \qquad G_2(s)=\frac{\Gamma(2s+4)\Gamma(s+3)}{\Gamma(1-s)}\Gamma\left(s+\frac{7}{3}\right) \tag{13}$$

It is easily verified that the functions

$$\Omega(s)=\frac{\Gamma(2s)\Gamma(s+1)}{\Gamma(2-s)}\Gamma\left(s+\frac{1}{3}\right), \qquad G_1(s), \qquad G_2(s)$$

satisfied the required conditions, namely, that their inverse Mellin transforms exist and that the original functions decay exponentially as x increases. Indeed, for arbitrary $c>0$ we have

$$\omega(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Omega(s)\frac{ds}{x^s}=8\int_0^\infty \exp-\frac{1}{z^4}J_2(2x^{1/4}z)K_2(2x^{1/4}z)\frac{dz}{z^{7/3}} \tag{14}$$

where $J_2(x)$ is the Bessel function of the first kind and second order, and $K_2(s)$ is Macdonald's function of the second order. The convergence of the integrals in question is evident.

Let us now determine the growth of $\omega(x)$ as x tends to infinity. Let us denote by M the maximum absolute value of $J_2(x)$ on $0\leqslant x<\infty$. From (14) we obtain

$$|\omega(x)|\leqslant 8M\int_0^\infty \exp\frac{-1}{z^4}K_2(2x^{1/4}z)\frac{dz}{z^{7/3}}\leqslant 8M\exp\left(\frac{3}{2\sqrt{2}}\right)\int_0^\infty \exp\left(-\frac{1}{z^4}\right)K_2(2x^{1/4}z)\frac{dz}{z^{7/3}} \tag{15}$$

Since x is large, and small z do not have a large influence in the last integral, we may replace $K(2x^{1/4}z)$ there by its asymptotic expansion. From this it follows that as $x \to \infty$ this integral decays like

$$\approx 2^{19/6}\pi M \exp \frac{3}{2\sqrt{2}}\, x^{1/24} \exp\left(-2^{3/2}x^{1/8}\right)$$

which, together with (15), yields the desired behavior for $\omega(x)$.

The corresponding considerations for the functions $G_1(s)$ and $G_2(s)$ can be carried out analogously.

Let us seek the function $\Psi(x)$ in the form of an asymptotic series

$$\psi(x) \approx \sum_{n=0}^{\infty} \frac{c_n}{x^{an+b}}$$

From Equation (12) we obtain $a = 2$, $b = 8/3$, and the following recurrence relation for the coefficients c_n:

$$c_{n+1} = \frac{3}{4}\Gamma\left(\frac{1}{3}\right) D\left(2n + \frac{8}{3}\right) c_n \qquad (n = 0, 1, \ldots), \qquad c_0 = -\frac{q}{4}\frac{\Gamma(1/3)}{\Gamma(16/3)\,\Gamma(11/3)}$$

where

$$D(s) \equiv \frac{G_1(s)}{G_2(s)} = -\frac{1}{2^4}\frac{1}{p(s)}\frac{\Gamma(-s)}{\Gamma(4/3 - s)} \tag{16}$$

$$p(s) = (s+2)\left(s+\frac{3}{2}\right)\left(s+\frac{4}{3}\right)(s+1)^2\left(s+\frac{1}{2}\right)\left(s+\frac{1}{3}\right)$$

Finally, for the unknown function $\Phi(x)$ we obtain from (11)

$$\varphi(x) \simeq \sum_{n=0}^{\infty} \frac{\Gamma(4n + 16/3)\,\Gamma(2n + 11/3)\,\Gamma(2n+3)}{\Gamma(-2n - 2/3)}\,\frac{c_n}{x^{2n+8/3}} \tag{17}$$

In conclusion, we remark that for many integral equations of the class being considered, which occur in applications, by combining functions of the particular form

$$\frac{\Gamma(2s)\,\Gamma(s+\nu)}{\Gamma(1-s+\nu)}\,\Gamma(s+\lambda_1)\ldots \quad \Gamma(s+\lambda_k) \qquad (\nu, \lambda_i \geqslant 0)$$

and their products one may construct functions $\Omega(s)$ (see (6) and (11) above) possessing the desired properties.

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